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TARGET LOCATION AND ID FROM A PASSIVE MULTISTATIC SENSOR NETWORK USING TIME DIFFERENCES OF ARRIVAL (TDOAs) AND THE HOUGH TRANSFORM

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WARFARE SYSTEMS DEPARTMENT

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Given a passive multistatic sensor network consisting of a set of commercial FM or acoustic transmitters, receivers, and a data processing center with their coordinate locations, scattered radiation responses can be used to identify fixed targets made up of straight lines and circular and elliptical arcs by employing the Hough Transform (HT) and its extensions. The HT is defined and examples of its use are given.

In order to apply the HT, a set of input points (x, y) is needed that is generated from transmitter responses, from a target to receivers, using time differences of arrival (TDOAs). Two methods are given to find these coordinates, primarily a straightforward approach dealing directly with the nonlinearity of the problem, and a particular application of Chan's method.

Fortran 95 software has been developed to utilize the procedures noted above.

14. SUBJECT TERMS
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FOREWORD

The analyses described in this report were carried out following discussions with William Ormsby of W12 and Jack Carr of Orca Computer Inc. ([1], [2]) addressing the need for software to support the passive multistatic system that Jack has under development. The study of the passive multistatic system was suggested by William Ormsby and was supported by the Office of the Under Secretary of Defense, Acquisition Technology and Logistics (USD AT&L)/Coalition Initiative (CI).

The author is indebted to Jack Carr for helpful discussions and some of the references, as well as proposing use of the Hough Transformation. William Ormsby reviewed the report and helped clarify the presentation.

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[1] MATLAB .m files to generate plots.

GLOSSARY

The numeral at the end of each item is the page number where the item is first used.

$\theta\rho$p– $\theta\rho$-plane, 3

A,a,B,b,D– Five parameters specifying an ellipse, 16

a,b,r– Parameters describing a circle, 10

$\mathrm{AO}, \mathrm{BO}, \mathrm{RO}$– Parameters that describe a circle, 14

atan2– Fortran subroutine for the four quadrant arctangent, 4

Chan– Chan's method, [4] of the main text, 1

CIR1– Subroutine that generates $\mathrm{a}, \mathrm{b}, \mathrm{r}$ from three given points, 11

CIRSOR– Subroutine extracting Mns on an $\mathrm{L-arc}$, 16

D– Determinant of linear system (27-28), 11

GUVM– Subroutine that generates the matrices $\{\rho_{\mathrm{ij}}\}$, $\{\theta_{\mathrm{ij}}\}$, 6

HSORT1– Subroutine that generates the output Mns on $\mathrm{L-lines}$, 6

HSORT2– Subroutine that extracts Mns on a circular arc with one point given, 14

HT– Hough Transform, 3

$\mathrm{L-arc}$– Circular arc. Generated by HSORT2, 10

$\mathrm{L-lines}$– Straight lines in xyp containing more than two Mns, 4

L– Straight line in the xy-plane, 2

Lk– the kth $\mathrm{L-line}$, $\mathrm{k} = 1, 2, \ldots, \mathrm{K}$, 4

M– An input set of N (x, y) points, 3

MCARR.FOR– File containing HSORT1, 6

MCARR1.FOR– File containing HSORT1, 6

Mn– A output point belonging to an $\mathrm{L-line}$ or an $\mathrm{L-arc}$, 3

N– A straight line normal to L and passing through the origin, 2

N– Number of (x, y) points, 1

NC– Number of input points on a circular arc, 11

Nk– Number of Mns in Lk with $\mathrm{N1} \geq \mathrm{N2} \geq, \ldots, \mathrm{NK} \geq 3$, 4

NL– Number of input points on a straight line, 11

NR– Number of input random points, 11

randn– Matlab subroutine which generates normal random numbers, 8

RNOR– Subroutine generates normal random numbers, 12

T– Denotes $\mathrm{B}^2 - 4\mathrm{A}$; A, B elliptic parameters, 17

TDOA– Time difference of arrival, 1

xyp– xy-plane, 2

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I. INTRODUCTION

The need for the analysis presented in this report is perhaps best described by Bill Ormsby (W11) and Jack Carr (ORCA Computer Inc.): "An acoustic, passive multistatic network (A-PMSN) for situation awareness in support of military and homeland security operations, such as harbor defense, is needed. The A-PMSN is a prototype sound source location system initially focused on the detection of small boats and the integration of those detections into existing track management procedures and operational picture displays. A-PMSN is designed to operate within a service-oriented architecture for early recognition of significant acoustic detections obtained from sensors located at known endpoints of a baseline using time differences of arrivals (TDOAs) of signals. A TDOA is defined to be the difference of times of arrival of a signal at two separated receivers (sensors).

Using sensors on more than one independent baseline or on one baseline with a track constraint allows the estimation of the positions of the sources of sounds being detected. These positions may include those for targets of interest and those for sources of sound not of interest. The TDOAs will be produced at regular time intervals and converted to estimated positions. These positions will be used as detections and processed by an associated cross correlation algorithm to produce tracks. The tracks are the result of the targets moving from fixed point to fixed point over the span of the data collection. In particular, targets whose positions fall close to a line and/or close to the arc of a circle are of particular interest and form the basis for the associated algorithm using the Hough Transform for preliminary estimates of target positions."

With a passive multistatic sensor network consisting of a set of commercial FM or acoustic transmitters, receivers (sensors), and a data processing center with their coordinate locations, one is interested in locating targets moving in straight lines and/or in circular or possibly elliptical arcs within a given surveillance area. An appropriate tool to aid in detecting preliminary estimates of such movements is the Hough Transform. In order to apply the Hough Transform (HT), N input points (x, y) are needed that are generated by transmitter responses from a target to receivers (sensors) using TDOAs [3].

Also, two methods are described to determine the (x, y) input points. First, a straightforward approach dealing directly with the nonlinearity of the problem is given, followed by a particular application of Ho-Chan's method [4].

In Section II, the HT is defined and discussed. In Section III the HT is used to identify targets made up of straight lines in the plane. Sections IV and V describe how extensions of the HT can be used to identify targets made up of circular and elliptical arcs, respectively. As noted above, the required input is a set of possible target Cartesian coordinates. Numerical examples are given.

Section VI addresses two procedures for finding the possible target coordinates, required by the HT and its extensions, using TDOAs. Three numerical examples are discussed.

Fortran 95 software has been developed to utilize the procedures noted above.

II. HOUGH TRANSFORM

Consider a straight line L in the xy-plane (xyp) with slope m, say

$$\mathrm{y} = \mathrm{m}\,(\mathrm{x} - \mathrm{x_o}\,) + \mathrm{y_o}\,, \qquad \mathrm{m} = \tan\phi, \quad (\text{See Figure}\,1), \tag{1}$$

where $(\mathrm{x_o},\ \mathrm{y_o})$ refers to a point on L. Let N denote the straight line that passes through the origin and is perpendicular to L at a unique $(\mathrm{x_o},\ \mathrm{y_o})$ as shown in Figure 1. Equation (1), can be written in terms of the parameters that describe N, namely θ and ρ. The result is

$$\mathrm{y}\sin\theta\, + \mathrm{x}\cos\theta\, = \rho, \quad -\pi \le \theta \le \pi, \tag{2}$$

where θ denotes the angle N makes with the positive x-axis, and ρ specifies the distance along N from (0, 0) to $(\mathrm{x_o},\ \mathrm{y_o})$. If in (1), $(\mathrm{x_o},\ \mathrm{y_o})\ = (0, 0)$, then $\rho = 0$.

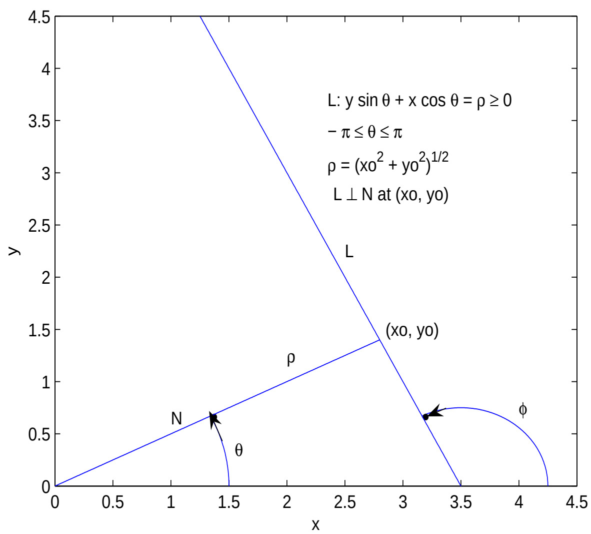

Figure 1. Normal Form of the Equation for a Straight Line

Equation (2) is referred to as the “normal equation for L.” It is obtained from (1) by observing that

$$\phi = \theta \pm \pi/2 \rightarrow \mathrm{m} = \tan\phi = -\cos\theta/\sin\theta\,. \tag{3}$$

Hence (1) becomes

$$\mathrm{y} = -(\cos\theta\,/\sin\theta\,)(\mathrm{x} - \mathrm{x_o}\,) + \mathrm{y_o} \tag{4}$$

$$\mathrm{y}\sin\theta\, + \mathrm{x}\cos\theta\, = \mathrm{x_o}\,\cos\theta\, + \mathrm{y_o}\,\sin\theta\,, \tag{5}$$

where

$$\rho = \mathrm{x_o}\,\cos\theta\, + \mathrm{y_o}\,\sin\theta\, = \pm\sqrt{\mathrm{x_o^2} + \mathrm{y_o^2}}\,, \qquad (\text{See Figure1}). \tag{6}$$

It may happen in evaluating ρ from (2) that $\rho < 0$, in which case π is added to θ if $\theta < 0$, or π is subtracted from θ if $\theta > 0$, thus reversing the sign of ρ. If $\rho = 0$ then π is added to θ if $\theta < 0$. Hence, it is assumed that ρ is always nonnegative. Note, (2) has the advantage over (1) since no problems are encountered at ϕ in the vicinity of $\pm\pi/2$.

Consider an input set of N *numerically* different points in xyp : $\mathrm{M} = \{\mathrm{Mn}\} = \{(\mathrm{x_n},\, \mathrm{y_n});\ \mathrm{n} = 1, \ldots, \mathrm{N}\}$. Hereafter, elements of M will be designated as *Mn points.* Let an L−line denote a straight line in xyp containing at least three Mn points. The HT transforms L to a point $(\theta,\, \rho)$ in the $\theta\rho$-plane ($\theta\rho$p), where the point is taken from the normal form of the equation for L as described above. At each Mn, a spectrum of straight lines can be drawn by varying θ in (2). So, each such Mn generates a curve in $\theta\rho$p. If this is done for each of the N points, the intersection points of the resulting curves in $\theta\rho$p designate which Mns belong to which L−lines, thus isolating the straight lines that can be generated from the N given points. Of course, one would look for three or more intersecting curves at a point in $\theta\rho$p. Figures 2 and 3 illustrate the result for six given Mns. Figure 3 exhibits $\rho < 0$ and applicable $\rho \geq 0$.

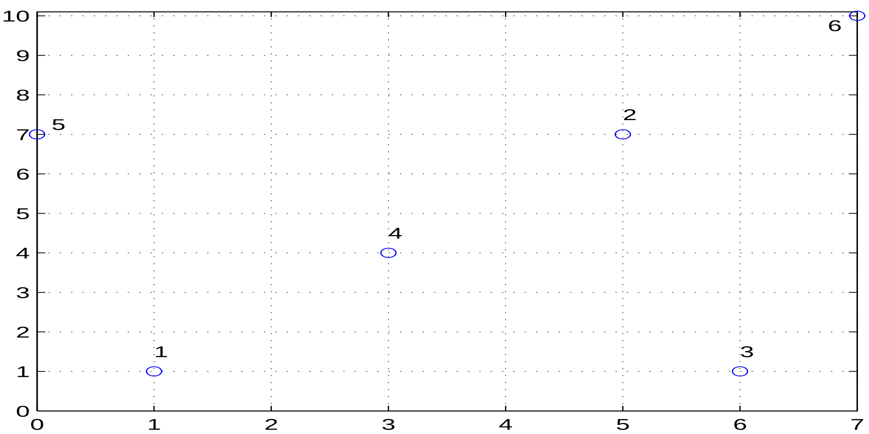

Figure 2. Given Mns, $(\mathrm{x_n},\, \mathrm{y_n})$, $\mathrm{n} = 1, \ldots, 6$, in xyp

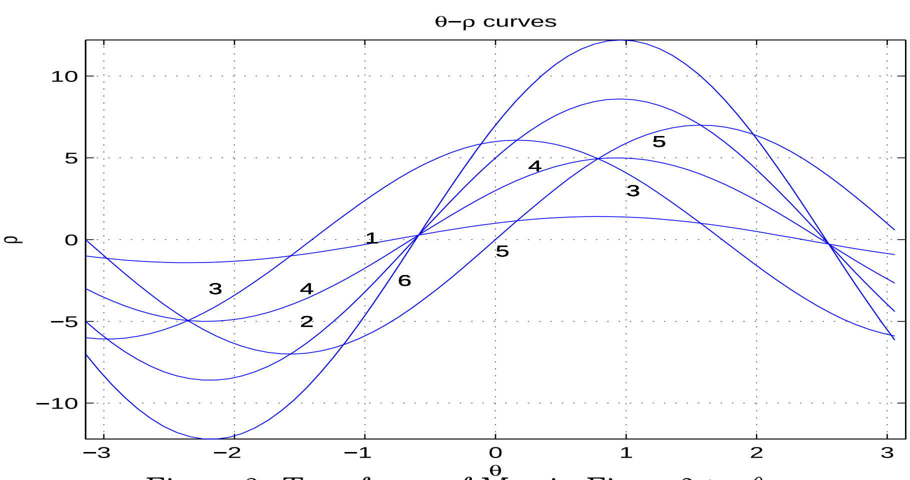

Figure 3. Transforms of Mns in Figure 2 to $\theta\rho$p

Thus, the HT can be used to isolate sequences of points that lie on straight lines embedded in a set M of discrete points (x_n, y_n), $n = 1, \ldots, N$.

The graphical approach, of using the curve intersections in $\theta\rho$p to find the L−lines, i.e., lines in the xyp containing more than two Mns, has the problem, if N is large, that the curve intersections in $\theta\rho$p are obscure; hence, difficult to read. In such cases, if one wanted the L−lines containing a large number of Mns, it could be difficult to separate them graphically. In addition, keep in mind that every three Mns generate an intersection of two curves in the $\theta\rho$p.

Our approach, which uses no graphing, is to begin by finding and itemizing the Mns belonging to each L−line, starting with the line(s) with the largest number of Mns and listing the remaining lines in decreasing order of their number of Mns. More precisely, let Lk denote the kth L−line obtained from the HT with $k = 1, 2, \ldots, K$, and let Nk denote the total number of Mns contained in Lk. Moreover, the K Lks are ordered in decreasing order of Nk, such that $N1 \geq N2 \geq, \ldots, NK \geq 3$.

In Figure 2, the Mns are indicated by small numbered circles. Table 1 lists the coordinates of these points.

Table 1. Listing of Mns of Figure 2

$$\begin{array}{|lcl|} [x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6\,] & = & [1,\ 5,\ 6,\ 3,\ 0,\ 7\,] \\ [y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6\,] & = & [1,\ 7,\ 1,\ 4,\ 7,\ 10\,] \end{array}$$

Thus, the third Mn would be $M3 = (x_3,\ y_3) = (6,\ 1)$.

Using Figures 2 and 3 as an example, there will clearly be two Lk lines, K = 2. So, the output would be as shown in Table 2.

Table 2. Listing Output Two L−lines

L1 →	N1 = 4,	Mns on L1 →	M1, M2, M4, M6
L2 →	N2 = 3,	Mns on L2 →	M3, M4, M5

The actual mechanics for generating software to determine the L−lines, Lk, with their Mns, is explained below using the example. Every two Mns determine a straight line with an associated unique θ and ρ that satisfy (2); then a θ and ρ are determined systematically for all possible pairs of Mns. Those pairs (at least two) that have the same θ and ρ determine an L − line passing through those Mns (as discussed with the intersections of curves in $\theta\rho$p where a θ and ρ are associated with each intersection). Beginning the process, Mns one and two would generate $\theta_{1,2}$ and $\rho_{1,2}$, and Mns i and j would generate $\theta_{i,j}$ and $\rho_{i,j}$ with $i < j$. In matrix notation, let $(\theta_{i,j})$ and $(\rho_{i,j})$ denote upper triangular matrices. We use a four-quadrant inverse tangent routine, atan2(y, x), such that $-\pi <$ atan2(y, x) $\leq \pi$. Throughout, the subscripts i, j are constrained to

$$\mathrm{i} < \mathrm{j},\ \mathrm{i} = 1, 2, \ldots, (\mathrm{N}-1),\ \mathrm{j} = 2, 3, \ldots, \mathrm{N}. \tag{7}$$

With ϵ discussed at the end of this section, the matrix elements are generated accordingly: If $|\mathrm{x_i} - \mathrm{x_j}| < \epsilon \max(|\mathrm{x_i}|, |\mathrm{x_j}|, 1)$ and $|\mathrm{y_i} - \mathrm{y_j}| \geq \epsilon \max(|\mathrm{y_i}|, |\mathrm{y_j}|, 1)$, then

$$\rho_{\mathrm{i,j}} = |\mathrm{x_i}| \quad \text{and} \quad \theta_{\mathrm{i,j}} = \begin{cases} 0, & \text{if } \mathrm{x_i} \geq 0, \\ \pi, & \text{if } \mathrm{x_i} < 0. \end{cases} \tag{8}$$

Else, if $|\mathrm{y_i} - \mathrm{y_j}| < \epsilon \max(|\mathrm{y_i}|, |\mathrm{y_j}|, 1)$ and $|\mathrm{x_i} - \mathrm{x_j}| \geq \epsilon \max(|\mathrm{x_i}|, |\mathrm{x_j}|, 1)$, then

$$\rho_{\mathrm{i,j}} = |\mathrm{y_i}| \quad \text{and} \quad \theta_{\mathrm{i,j}} = \begin{cases} \pi/2, & \text{if } \mathrm{y_i} \geq 0, \\ -\pi/2, & \text{if } \mathrm{y_i} < 0. \end{cases} \tag{9}$$

Else, if $(|\mathrm{x_i} - \mathrm{x_j}| \geq \epsilon \max(|\mathrm{x_i}|, |\mathrm{x_j}|, 1)$ and $|\mathrm{y_i} - \mathrm{y_j}| \geq \epsilon \max(|\mathrm{y_i}|, |\mathrm{y_j}|, 1))$, then

$$\overline{\theta}_{\mathrm{i,j}} = \begin{cases} \tan^{-1}[\,(\mathrm{x_i} - \mathrm{x_j})/(\mathrm{y_j} - \mathrm{y_i})\,], & \mathrm{i} < \mathrm{j}, \\ 0, & \mathrm{i} \geq \mathrm{j}, \end{cases} \tag{10}$$

$$\overline{\rho}_{\mathrm{i,j}} = \begin{cases} \mathrm{x_i} \cos \overline{\theta}_{\mathrm{i,j}} + \mathrm{y_i} \sin \overline{\theta}_{\mathrm{i,j}}, & \mathrm{i} < \mathrm{j}, \\ 0, & \mathrm{i} \geq \mathrm{j}. \end{cases} \tag{11}$$

If $\overline{\rho}_{\mathrm{i,j}} > 0$, then

$$\rho_{\mathrm{i,j}} = \overline{\rho}_{\mathrm{i,j}}, \quad \theta_{\mathrm{i,j}} = \overline{\theta}_{\mathrm{i,j}}, \tag{12}$$

else, if $\overline{\rho}_{\mathrm{i,j}} = 0$, then

$$\rho_{\mathrm{i,j}} = \overline{\rho}_{\mathrm{i,j}} \text{ and } \theta_{\mathrm{i,j}} = \begin{cases} \overline{\theta}_{\mathrm{i,j}}, & \text{if } \overline{\theta}_{\mathrm{i,j}} \geq 0, \\ \overline{\theta}_{\mathrm{i,j}} + \pi, & \text{if } \overline{\theta}_{\mathrm{i,j}} < 0, \end{cases} \tag{13}$$

else, if $\overline{\rho}_{\mathrm{i,j}} < 0$, then

$$\rho_{\mathrm{i,j}} = -\overline{\rho}_{\mathrm{i,j}} \text{ and } \theta_{\mathrm{i,j}} = \begin{cases} \overline{\theta}_{\mathrm{i,j}} + \pi, & \text{if } \overline{\theta}_{\mathrm{i,j}} \leq 0, \\ \overline{\theta}_{\mathrm{i,j}} - \pi, & \text{if } \overline{\theta}_{\mathrm{i,j}} > 0. \end{cases} \tag{14}$$

Referring to the example involving Figures 2 and 3, we obtain

$$\{\,\theta_{\mathrm{i,j}}\,\} = \begin{pmatrix} 0 & -.5880 & 1.5708 & -.5880 & .1651 & -.5880 \\ 0 & 0 & .1651 & -.5880 & 1.5708 & -.5880 \\ 0 & 0 & 0 & .7854 & .7854 & -.1107 \\ 0 & 0 & 0 & 0 & .7854 & -.5880 \\ 0 & 0 & 0 & 0 & 0 & 1.9757 \end{pmatrix} \tag{15}$$

$$\{\,\rho_{\mathrm{i,j}}\,\} = \begin{pmatrix} 0 & .2774 & 1 & .2774 & 1.1508 & .2774 \\ 0 & 0 & 6.0828 & .2774 & 7 & .2774 \\ 0 & 0 & 0 & 4.9497 & 4.9497 & 5.8527 \\ 0 & 0 & 0 & 0 & 4.9497 & .2774 \\ 0 & 0 & 0 & 0 & 0 & 6.4340 \end{pmatrix} \tag{16}$$

Note that $\overline{\theta}_{5,6} = \tan^{-1}[(0-7)/(10-7)] = -1.165904$ with $\overline{\rho}_{5,6} = y_5 \sin(-1.165904) = -6.43402$. Thus, (14) holds and to insist on $\rho_{5,6} > 0$, $\rho_{5,6} = -\overline{\rho}_{5,6}$ and π is added to $\overline{\theta}_{5,6}$, so that $\theta_{5,6} = \overline{\theta}_{5,6} + \pi = 1.9757$, as noted in (15). The software which generates the matrices in (15) and (16) is the subroutine GUVM.

The software designed to determine the Lk lines and their associated Mns is Fortran subroutine HSORT1 contained in file H:\CARR\MCARR.FOR or H:\CARR\MCARR1.FOR. The reasoning behind the software is explained using (8 – 16). Starting with $\theta_{1,2}$, one looks along the first row for $\theta_{1,j} = \theta_{1,2}$ and equality in the corresponding $\rho_{1,j} = \rho_{1,2}$. If such are found for $j \geq 3$, then one need look no further; an Lk has been found. From (12 – 13), we have $\theta_{1,2} = \theta_{1,4} = \theta_{1,6} = -.5880$ and $\rho_{1,2} = \rho_{1,4} = \rho_{1,6} = .2774$. All the subscripts indicate the Mns as shown in Table 2. There is no need to examine subsequent rows, for if another row element occurs with the above equalities, its Mn will already have appeared in the subscripts obtained from the original row, in this case row one. For example $\theta_{2,4} = \theta_{2,6} = -.5880$ and $\rho_{2,4} = \rho_{2,6} = .2774$. We continue with searches in the first row of (15), but no more matches are found. In the third row we find two matches: $\theta_{3,4} = \theta_{3,5} = .7854$ and $\rho_{3,4} = \rho_{3,5} = 4.9497$ that yield the Mns given in second row of Table 2. The two Lks are ordered as noted above and listed in Table 2 as L1, L2.[2]

In HSORT1, numerical equalities for the elements $\theta_{i,j}$ and $\rho_{i,k}$ are defined in terms of an input epsilon, $\epsilon > 0$ (as also used in (8-11)), as

$$|\,\theta_{i,j} - \theta_{i,k}\,| \leq \max(1,\, |\,\theta_{i,j}\,|,\, |\,\theta_{i,k}\,|)\,\epsilon; \tag{17}$$

similarly for the elements $\rho_{i,j}$ and $\rho_{i,k}$, numerical equality occurs if

$$|\,\rho_{i,j} - \rho_{i,k}\,| \leq \max(1,\, \rho_{i,j},\, \rho_{i,k}\,)\,\epsilon. \tag{18}$$

Two references with an extensive discussion of HT are [6], [7].

III. TARGET ID USING THE HOUGH TRANSFORM

Two more complicated applications of HT are discussed here. In the first case, we consider 16 Mn points, as shown in Figure 4, and obtain L-lines using HSORT1. The input points, Mns, are also given in Table 3.

Table 3. Listing of Mns of Figure 4

$[x_1, x_2, \ldots, x_{16}]$	=	[1	5	6	3	0	7	3	3	0	5	3	7	5	0	2	5]
$[y_1, y_2, \ldots, y_{16}]$	=	[1	7	1	4	7	10	1	7	0	2	3	7	5	10	0	0]

[2]Speed of computation was necessary for HSORT1; a faster intricate program, written by Russ Gnoffo, can be used instead of HSORT1 if needed.

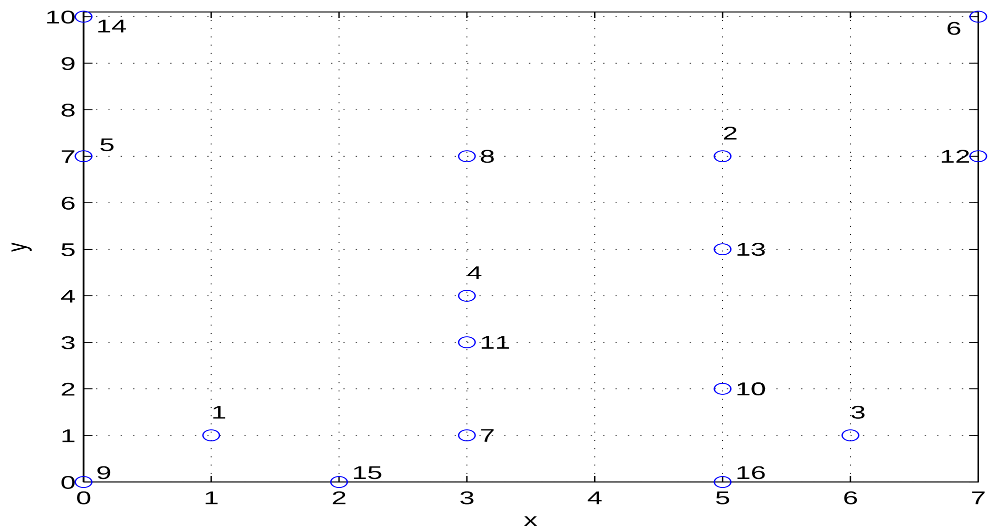

Figure 4. Given Mns, $(\mathrm{x_n},\ \mathrm{y_n})$, $\mathrm{n} = 1, \ldots, 16$, in xyp

The corresponding curves in $\theta\rho$p for the Mns are shown in Figure 5. Note the relative difficulty in reading the intersection points in $\theta\rho$p .

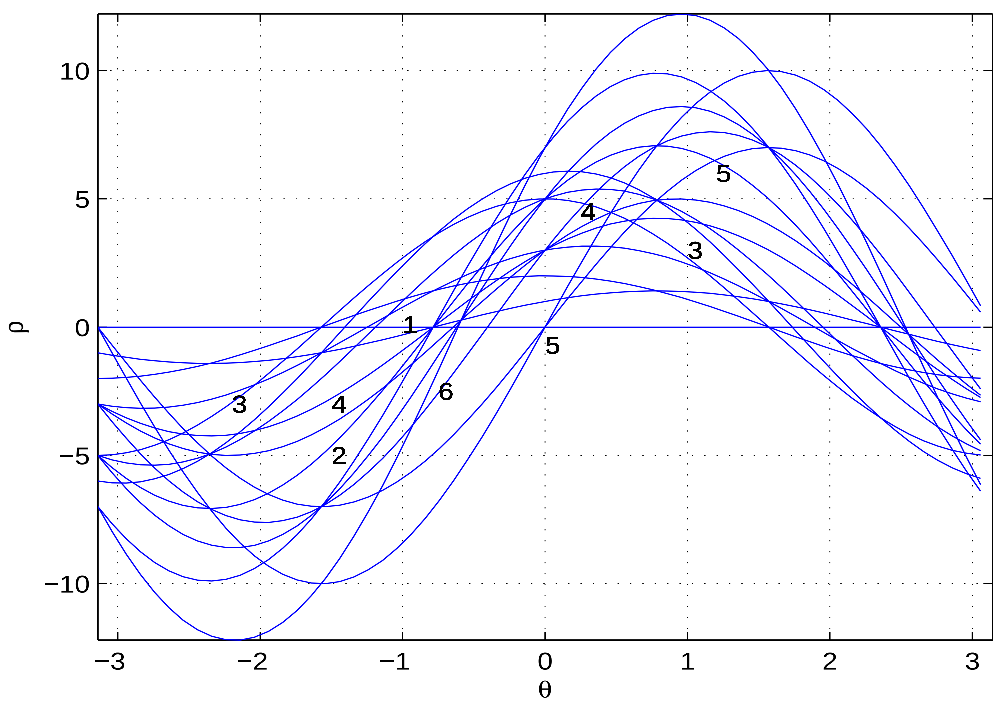

Figure 5. Transforms of the Mns in Figure 4 to $\theta\rho$p

The Lk lines determined from the Mns given in Table 3, using HSORT1, are given in Table 4.

Table 4. Listing Output Eleven L−lines From Table 3 Input

L1 $\rightarrow$	N1 = 5,	Mns on L1 $\rightarrow$	$1, 9, 11, 12, 13$
L2 $\rightarrow$	N2 = 4,	Mns on L2 $\rightarrow$	$1, 2, 4, 6$
L3 $\rightarrow$	N3 = 4,	Mns on L3 $\rightarrow$	$4, 7, 8, 11$
L4 $\rightarrow$	N4 = 4,	Mns on L4 $\rightarrow$	$3, 4, 5, 10$
L5 $\rightarrow$	N5 = 4,	Mns on L5 $\rightarrow$	$2, 10, 13, 16$
L6 $\rightarrow$	N6 = 4,	Mns on L6 $\rightarrow$	$2, 5, 8, 12$
L7 $\rightarrow$	N7 = 3,	Mns on L7 $\rightarrow$	$1, 3, 7$
L8 $\rightarrow$	N8 = 3,	Mns on L8 $\rightarrow$	$4, 14, 16$
L9 $\rightarrow$	N9 = 3,	Mns on L9 $\rightarrow$	$8, 13, 14$
L10 $\rightarrow$	N10 = 3,	Mns on L10 $\rightarrow$	$5, 9, 14$
L11 $\rightarrow$	N11 = 3,	Mns on L11 $\rightarrow$	$9, 15, 16$

In order to simplify the notation, we have dropped the M prefix on the Mn points, so that, for example, M1, M9, M11, M12, M13 for L1 appear as noted in Table 4.

The second application of this section involves extracting the outline of a house, composed of straight lines, emersed in a maze of NR = 350 random points. The house outline is made up of thirty Mns that determine five L-lines. The points are given by:

$$\begin{cases} \mathrm{x} &= 1.0, \quad \mathrm{y} = 1.0,\ 1.2,\ 1.4,\ 1.6,\ 1.8 \\ \mathrm{y} &= \mathrm{x}+1, \quad \mathrm{x} = 1.0,\ 1.2,\ 1.4,\ 1.6,\ 1.8,\ 2.0 \\ \mathrm{y} &= 5-\mathrm{x}, \quad \mathrm{x} = 2.2,\ 2.4,\ 2.6,\ 2.8 \\ \mathrm{x} &= 3.0, \quad \mathrm{y} = 2.0,\ 1.8,\ 1.6,\ 1.4,\ 1.2 \\ \mathrm{y} &= 1.0, \quad \mathrm{x} = 3.0,\ 2.8,\ 2.6,\ 2.4,\ 2.2,\ 2.0,\ 1.8,\ 1.6,\ 1.4,\ 1.2 \end{cases} \tag{19}$$

The NR random points are taken from a Matlab routine based on normal distributions. Therefore, using Matlab code:

$$\begin{cases} \mathrm{mn} = 350; \\ \mathrm{randn}('\mathrm{seed}', 1); \\ \mathrm{x} = .55 + 2.35 * \mathrm{abs}(\mathrm{randn}(1, \mathrm{mn})); \\ \mathrm{randn}('\mathrm{seed}', 2); \\ \mathrm{y} = .50 + 1.75 * \mathrm{abs}(\mathrm{randn}(1, \mathrm{mn})); \end{cases} \tag{20}$$

Figure 6 shows a plot of the 30 Mns and those random points in the interval $0.5 < \mathrm{x} \leq 3.5$. It is certainly difficult to identify the house outline (30 Mns) in Figure 6. Figure 7 shows the same points with the Mn points darkened.

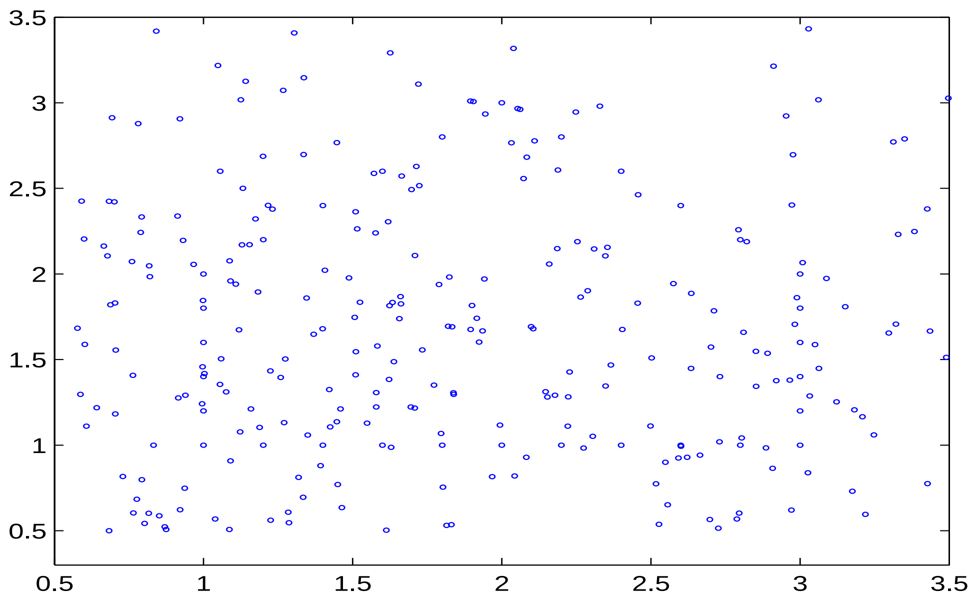

Figure 6. A Plot of 30 Mns and a Subset of 350 Random Points

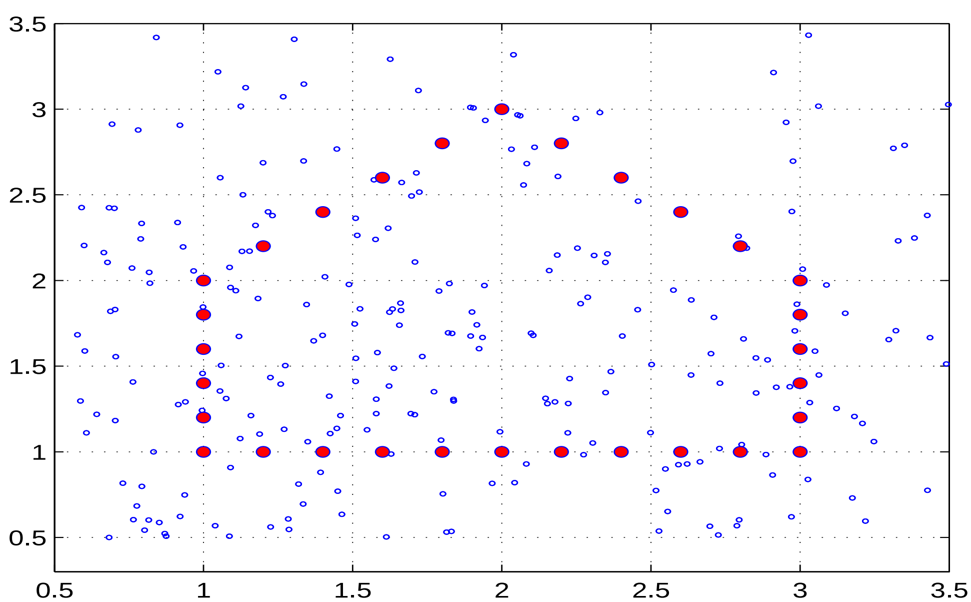

Figure 7. A Plot of the Identical Data of Figure 6 with the 30 Mns Darkened

The input array (x_i, y_i), $i = 1, \ldots, 380$ was arranged with the 30 Mns making up the first 30 points of the array, followed by the NR random points, but the results, of course, are independent of the order in which the input points are given. This is so because every possible pair of points in the input array is treated.

Calling on the Fortran subroutine HSORT1 to find all L-lines containing four or more Mns, using as input the (x, y) of (19) and (20), the output is given in Table 5.

Table 5. Listing Output from Input Given by (19) and (20)

L1 $\rightarrow$	N1=11,	Mns on L1 $\rightarrow$	$1, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30$
L2 $\rightarrow$	N2=6,	Mns on L2 $\rightarrow$	$11, 12, 13, 14, 15, 16$
L3 $\rightarrow$	N3=6,	Mns on L3 $\rightarrow$	$6, 7, 8, 9, 10, 11$
L4 $\rightarrow$	N4=6,	Mns on L4 $\rightarrow$	$1, 2, 3, 4, 5, 6$
L5 $\rightarrow$	N5=6,	Mns on L5 $\rightarrow$	$16, 17, 18, 19, 20, 21$

IV. EXTENSION OF THE HOUGH TRANSFORM TO CIRCULAR ARCS

In the previous section, using HT, it was shown that Mns to determine L-lines could be extracted from a set M of points in xyp. The principle involved is simply that any straight line in xyp can be defined by two parameters (θ, ρ in the HT case), so that examining every pair of points in M and finding those pairs with the same θ, ρ establishes Mns that determine an L-line.

In the case of circular arcs, we use the fact that a circle in the plane is uniquely defined by three parameters. Therefore, to determine an L−arc , i.e., an arc of a circle (corresponding to a straight line in HT case), one would look at all possible combinations of three points from M and search for those particular three-point combinations that have the same values for each of the three parameters. The Fortran software that accomplishes this is contained in file DH.FOR. Subprogram DGXY generates M-sets for testing.

We use for the three parameters defining a circle, the center of the circle, (a, b), and its radius r. Thus, the equation for a circle in xyp is

$$(\mathrm{x} - \mathrm{a})^2 + (\mathrm{y} - \mathrm{b})^2 = \mathrm{r}^2. \tag{21}$$

The objective is to evaluate a, b, and r using three distinct points from M. The fact that a and b occur nonlinearly presents no problem. Denote the three points by (x1, y1), (x2, y2), (x3, y3). Substitute these quantities into (21) to obtain

$$(\mathrm{x1} - \mathrm{a})^2 \quad + \quad (\mathrm{y1} - \mathrm{b})^2 = \mathrm{r}^2, \tag{22}$$

$$(\mathrm{x2} - \mathrm{a})^2 \quad + \quad (\mathrm{y2} - \mathrm{b})^2 = \mathrm{r}^2, \tag{23}$$

$$(\mathrm{x3} - \mathrm{a})^2 \quad + \quad (\mathrm{y3} - \mathrm{b})^2 = \mathrm{r}^2. \tag{24}$$

Then subtracting (22) from (23) and (22) from (24), one obtains, after cancelations:

$$2\,(\mathrm{x2} - \mathrm{x1})\,\mathrm{a} + 2\,(\mathrm{y2} - \mathrm{y1})\,\mathrm{b} \;=\; \mathrm{x2}^2 - \mathrm{x1}^2 + \mathrm{y2}^2 - \mathrm{y1}^2, \tag{25}$$

$$2\,(\mathrm{x3} - \mathrm{x1})\,\mathrm{a} + 2\,(\mathrm{y3} - \mathrm{y1})\,\mathrm{b} \;=\; \mathrm{x3}^2 - \mathrm{x1}^2 + \mathrm{y3}^2 - \mathrm{y1}^2. \tag{26}$$

Solving the last two equations for a and b gives

$$\begin{aligned} 2\,\mathrm{a} &= \left[(\mathrm{x2}^2+\mathrm{y2}^2)(\mathrm{y3}-\mathrm{y1})-(\mathrm{x1}^2+\mathrm{y1}^2)(\mathrm{y3}-\mathrm{y2})-(\mathrm{x3}^2+\mathrm{y3}^2)(\mathrm{y2}-\mathrm{y1})\right]/\mathrm{D}, \quad (27)\\ 2\,\mathrm{b} &= \left[(\mathrm{x2}^2+\mathrm{y2}^2)(\mathrm{x1}-\mathrm{x3})+(\mathrm{x1}^2+\mathrm{y1}^2)(\mathrm{x3}-\mathrm{x2})+(\mathrm{x3}^2+\mathrm{y3}^2)(\mathrm{x2}-\mathrm{x1})\right]/\mathrm{D}, \quad (28)\end{aligned}$$

where different algebraic expressions for D are given by:

$$\mathrm{D} = \begin{cases} (\mathrm{x2}-\mathrm{x1})(\mathrm{y3}-\mathrm{y1})-(\mathrm{x3}-\mathrm{x1})(\mathrm{y2}-\mathrm{y1}) & \neq \;0,\\ \mathrm{x1}(\mathrm{y2}-\mathrm{y3})+\mathrm{x2}(\mathrm{y3}-\mathrm{y1})+\mathrm{x3}(\mathrm{y1}-\mathrm{y2}) & \neq \;0,\\ \mathrm{y1}(\mathrm{x3}-\mathrm{x2})+\mathrm{y2}(\mathrm{x1}-\mathrm{x3})+\mathrm{y3}(\mathrm{x2}-\mathrm{x1}) & \neq \;0.\end{cases} \quad (29)$$

If $\mathrm{D} = 0$, then either the three input points lie on a straight line, or at least two of the points are the same. For either of these inputs, no circle is determined.

The remaining unknown r is obtained from any one of (22)-(24), using a and b obtained from (27) and (28).

The software that reflects this analysis is the subroutine $\mathrm{CIR1}(\mathrm{x1}, \mathrm{y1}, \mathrm{x2}, \mathrm{y2}, \mathrm{x3}, \mathrm{y3}, \mathrm{a}, \mathrm{b}, \mathrm{r}, \mathrm{D})$.

Hence, to extract the circular arcs from M, the subroutine CIRSOR generates three arrays $\mathrm{U}(\mathrm{i},\mathrm{j},\mathrm{k})$, $\mathrm{V}(\mathrm{i},\mathrm{j},\mathrm{k})$, and $\mathrm{R}(\mathrm{i},\mathrm{j},\mathrm{k})$ that are functions of *triplets* of all possible combinations of three points from M, without repetition (of which there are N!/[3!(N-3)!]). So that for a fixed $\mathrm{i},\mathrm{j},\mathrm{k}$ the element of each array is constructed by CIRSOR as:

$$\begin{aligned} \mathrm{U(i,\,j,\,k)} &= \mathrm{a}, & \mathrm{i} &= 1,\ldots,\mathrm{N}-2, & \mathrm{j} &= \mathrm{i}+1,\ldots,\mathrm{N}-1, & \mathrm{k} &= \mathrm{j}+1,\ldots,\mathrm{N},\\ \mathrm{V(i,\,j,\,k)} &= \mathrm{b}, & \mathrm{i} &= 1,\ldots,\mathrm{N}-2, & \mathrm{j} &= \mathrm{i}+1,\ldots,\mathrm{N}-1, & \mathrm{k} &= \mathrm{j}+1,\ldots,\mathrm{N}, \quad (30)\\ \mathrm{R(i,\,j,\,k)} &= \mathrm{r}^2, & \mathrm{i} &= 1,\ldots,\mathrm{N}-2, & \mathrm{j} &= \mathrm{i}+1,\ldots,\mathrm{N}-1, & \mathrm{k} &= \mathrm{j}+1,\ldots,\mathrm{N},\end{aligned}$$

where CIRSOR calls CIR1 to find the elements a, b, r^2 of U, V, R, respectively, corresponding to each triplet of input. If $\mathrm{D} = 0$, r is set to zero.

The similarity to (10) and (11) should be clear.

As a special case, file PTCIR.FOR contains Fortran software that finds $\mathrm{L-lines}$ each with an attached $\mathrm{L-arc}$ at an endpoint. As an example, we consider M made up of a set of NR=400 random points from normal distributions, a horizontal line L with NL=19 points, and a circular arc, $\mathrm{L-arc}$, with NC=10 points, that begins at the right end-point of L, $(\mathrm{y} = 3,\ 1 \le \mathrm{x} \le 3)$ and forms an arc of the circle: $(\mathrm{x}-1)^2 + (\mathrm{y}-4)^2 = 5$. Hence M contains a total of N=429 points in xyp, where the Fortran 95 code that follows indicates the generation of the NR=400 random points, followed by the NL=19 straight line points, and ends by generating the ten $\mathrm{L-arc}$ points. The code that generates these points is carried out by the subroutine GXYC, which follows.

```
      DIMENSION  X1(429), Y1(429)
      NR = 400
      NL = 19
      NC = 10
      IR = 1 !Seed for random number generator.
      IS = 2
      CALL RNOR(IR,X1,NR,IER) !Subroutine RNOR taken from [5].
      CALL RNOR(IS,Y1,NR,IES) !Random coordinates.

      DO 5  I = 1,NR
         X1(I) = 3.5*ABS(X1(I))
5        Y1(I) = 4. + Y1(I)

      AO = 1.
       BO1 = 3.
       DT = (BO1 - AO)/NL
       NR1 = NR + 1
       X1(NR1) = AO + DT
       Y1(NR1) = BO1
       DO 10  I = NR1, NR+NL - 1
         X1(I+1) = X1(I) + DT
10       Y1(I+1) = Y1(I)
       PI = 3.14159265359
       NRC = NR + NL + 1
       NCI = (NC - 1)*2
       PIN = PI/NCI
       BO = 4.
       RO = √5
       THETA = ATAN2((BO1-BO),(BO1-AO))

      DO 15  I = NRC, N
         THETA = THETA + PIN
         X1(I) = AO + RO*COS(THETA)
15       Y1(I) = BO + RO*SIN(THETA)
```

Figure 8 shows some of the random points and the points belonging to the straight line, but the points belonging to the arc, although included, are not evident. Figure 9 has the same points as Figure 8 with the straight line and circular arc points enhanced.

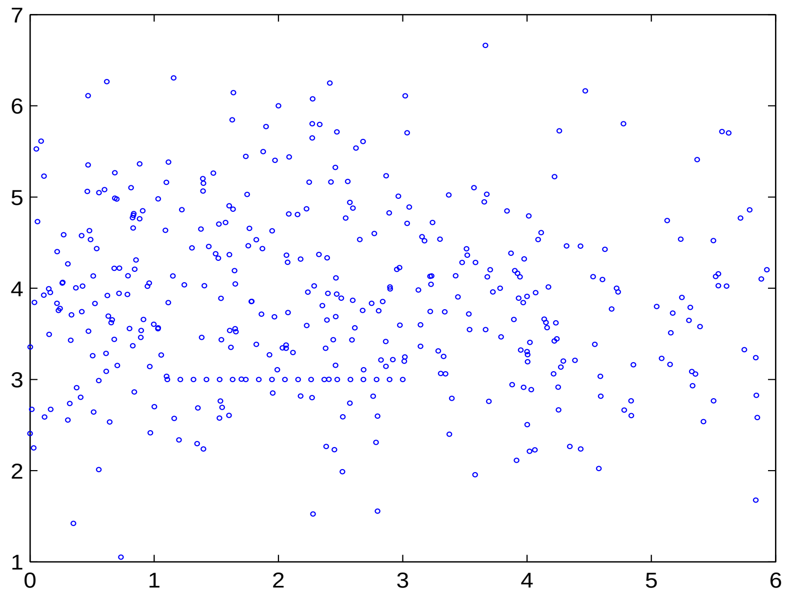

Figure 8. A Plot of Random Data and Some Points Forming a Straight Line with an Attached Circular Arc

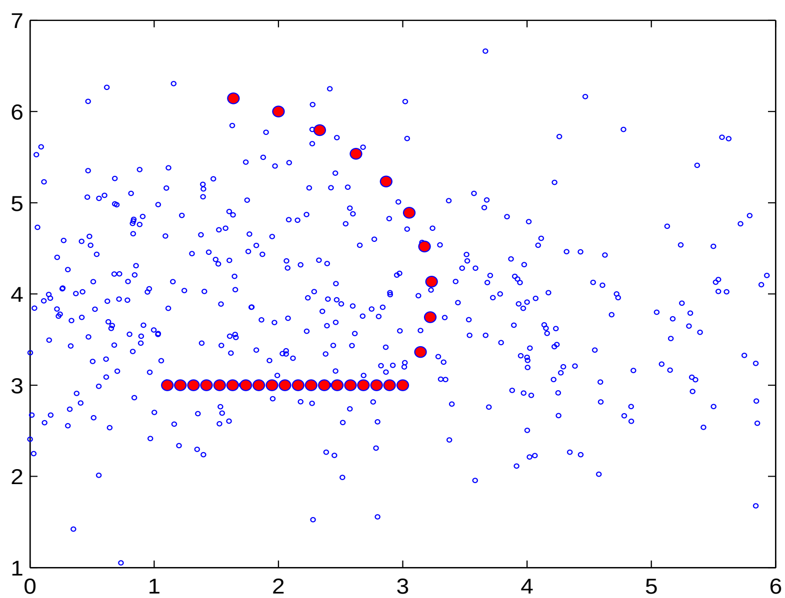

Figure 9. Same Plot as in Figure 8 with the Straight Line and Circular Arc Points Enhanced

The object of this simulation is now to extract the Mn points belonging to the L-line using subroutine HSORT1 and the attached L−arc using subroutine HSORT2. The results are given in Table 6.

Table 6. Listing Output Using HSORT1 and HSORT2

```
***DOUBLE PRECISION PTCIRC.FOR *** JM= 0, MZ= 4
NR = 400, NL = 19, NC = 10, N = 429, AO=1, BO=4, RO=2.23607
1 19 (The N points of M are input for HSORT1.)
401 402 403 404 405 406 407 408 409 410 411 412 413 414 415 416 417 418 419
****************************************
IX 419
1 10 (Only the N points of M and attached point 419 are input for HSORT2.)
420 421 422 423 424 425 426 427 428 429
2 6
123 154 189 239 291 298
3 6
53 180 190 233 306 329
```

In the first line of the table, JM is not relevant; MZ=4 indicates that only output with at least four Mn's will be given. The second line specifies the input for generating the test data. The third line notes the first line of output, L1, using HSORT1, and indicates 19 points. The fourth line lists the 19 points. The sixth line indicates the point at which the L-line and L–arc are attached. The seventh line gives the first line of output, with 10 points, using HSORT2, referring to the $\mathrm{L-arc}$. The eighth line specifically gives the 10 points. The last four lines refer to $\mathrm{L-arcs}$ contained in the distribution of random points and attached to point 419.

Note, if (xe,ye) denotes the last point of L, that all triplets of M chosen include (xe,ye). Hence, only every pair of points of M, without repetition, make up the input together with (xe,ye), rather than an independent triplet of (x,y) points. The software returns results more quickly under these conditions.

In the final example of this section, it is required to find points belonging to three circular arcs, three L–arcs, embedded in a set M also containing random points. This differs from the previous example where the $\mathrm{L-arc}$ was attached to the L-line. Thus, here the three points required to establish an $\mathrm{L-arc}$ are independent so that the size of the problem increases significantly from an N-squared problem, as in the previous example, to an N-cubed problem. We need to consider all possible combinations of the points of M taking three points at a time without repetitions. The total number of such combinations is NT=N(N-1)(N-2)/6 rather than N(N-1)/2 as in the previous example where all possible *pairs* were considered.

File DH.FOR contains subroutine DGXY that constructs the simulation by specifying the three $\mathrm{L-arcs}$; $\mathrm{L-ai, a = aoi, b = boi, r = roi, i = 1, 2, 3.}$ Thus

$$\begin{aligned} \mathrm{L-a1:}\quad & \mathrm{x = ao1 + ro1} * \cos\theta, \quad \mathrm{ao1 = 1, bo1 = 4, ro1 = 4} \\ & \mathrm{y = bo1 + ro1} * \sin\theta, \quad 30 \leq \theta \leq 100\,(\text{degrees}),\ 11 \text{ plotted points.} \end{aligned} \tag{31}$$

$$\begin{aligned} \mathrm{L-a2:}\quad & \mathrm{x = ao2 + ro2} * \cos\theta, \quad \mathrm{ao2 = 3, bo2 = 4, ro2 = 3} \\ & \mathrm{y = bo2 + ro2} * \sin\theta, \quad 30 \leq \theta \leq 150\,(\text{degrees}),\ 21 \text{ plotted points.} \end{aligned} \tag{32}$$

$$\begin{aligned} \mathrm{L-a3:}\quad & \mathrm{x = ao3 + ro3} * \cos\theta, \quad \mathrm{ao3 = -1, bo3 = 4, ro3 = 3.5} \\ & \mathrm{y = bo3 + ro3} * \sin\theta, \quad -60 \leq \theta \leq 60\,(\text{degrees}),\ 16 \text{ plotted points.} \end{aligned} \tag{33}$$

Figures 10 and 11 show the points of M within the x and y intervals of the graphs where $\mathrm{N} = 160$ and $\mathrm{NT} = \mathrm{N}\,(\mathrm{N}-1)\,(\mathrm{N}-2)/6 = 669920$.

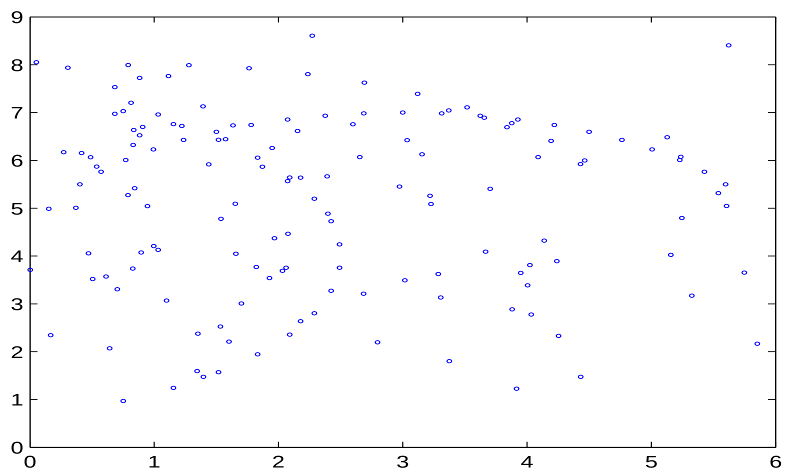

Figure 10. A Plot of Random Data and Points Forming Three Circular Arcs

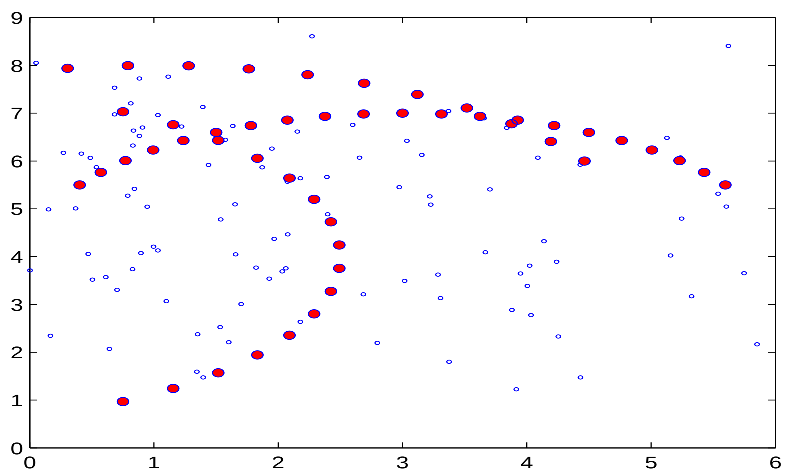

Figure 11. Same Plot as in Figure 10 with Circular Arc Points Enhanced

The extraction of the plotted points of the three L – arcs is carried out by the subroutine CIRSOR in file DH.FOR. In more detail, the generation of the simulated data using DOXY was done in the following order: First N (=160) random (x,y) points were generated and stored in array AB; then alternating starting with location AB(5), a point from each of the L – arcs specified in (31)-(33) was inserted into the AB overwriting those locations; so that $\mathrm{L-a1}$ is stored in $\mathrm{AB(5), AB(8), \ldots, AB(35)}$; $\mathrm{L-a2}$ is stored in $\mathrm{AB(6), AB(9), \ldots, AB(66)}$; $\mathrm{La-3}$ is stored in $\mathrm{AB(7), AB(10), \ldots, AB(52)}$.

The output form CIRSOR is shown in Table 7.

Table 7. Listing Output Using CIRSOR

NR = 112, NC = 48, N = 160. See (31)-(33) for L – arc points.
****
TIME 29,
L – a1-> ICN(1) = 11, IC = 5 8 11 14 17 20 23 26 29 32 35
L – a2-> ICN(2) = 21, IC = 6 9 12 15 18 21 24 27 30 33 36 39 42 45 48 51 54 57 60 63 66
L – a3-> ICN(3) = 16, IC = 7 10 13 16 19 22 25 28 31 34 37 40 43 46 49 52

V. EXTENSION OF THE HOUGH TRANSFORM TO ELLIPTICAL ARCS

We consider the equation for the ellipse E in the xy-plane given by

$$\mathrm{(x-a)^2 + A(y-b)^2 + Bxy + D = 0, \quad T \equiv B^2 - 4A < 0.} \tag{34}$$

There are five unknowns to determine: $\mathrm{a, b, A, B, D}$. We use five $\mathrm{(x, y)}$ points on E to evaluate the unknowns. Or, we can state the problem as: Given five points in the plane, what are the values of the unknowns above such that the five points are on E? Note that if the inequality in (34) is not satisfied, then other geometrical objects will be determined such as a hyperbola, parabola, or straight line. However, our emphasis will be with ellipses.

Let the five points be denoted by $\mathrm{(x_k,\ y_k),\ \ k = 1, ..., 5}$. Then four linear equations can be generated for $\mathrm{a, b, A, B}$ after $\mathrm{a^2}$ and $\mathrm{b^2}$ are eliminated. Namely

$$\mathrm{2a\,(x_k - x_1) + A\,(y_1^2 - y_k^2) + C\,(y_k - y_1) + B\,(x_1\,y_1 - x_k\,y_k) = x_k^2 - x_1^2, \quad k = 2, ..., 5,} \tag{35}$$

where

$$\mathrm{b = C/(2\,A).} \tag{36}$$

By using the results of solving the linear system (35) with (36), D is evaluated from (34) with, say, $\mathrm{x = x_1,\ y = y_1}$.

Example (calculations carried out in double precision):

Specify E by:

$$(\mathrm{x}-5)^2+\frac{50}{18}(\mathrm{y}-5)^2+2.5\mathrm{xy}-94\,\tfrac{4}{9}=0,\ \ \mathrm{T}\equiv\mathrm{B}^2-4\mathrm{A}=6.25-100/9<0. \tag{37}$$

Thus,

$$\mathrm{a}=5,\ \mathrm{b}=5,\ \mathrm{A}=50/18,\ \mathrm{B}=2.5,\ \mathrm{D}=-94\,\tfrac{4}{9}. \tag{38}$$

The objective is to get good numerical approximations to these quantities by assigning five points on E. Hence, using the five points on E, $(0,0)$, $(2,8.85078940)$, $(4,7.54511220)$, $(6,6.03229152)$, $(8,4.17848880)$, we obtain from (35) the linear system:

$$\begin{pmatrix} 2.00000000 & -78.3364731 & 8.85078940 & -17.7015788 \\ 4.00000000 & -56.9287181 & 7.54511220 & -30.1804488 \\ 6.00000000 & -36.3885410 & 6.03229152 & -36.1937491 \\ 8.00000000 & -17.4597686 & 4.17848880 & -33.4279104 \end{pmatrix}\begin{pmatrix} 2\mathrm{a} \\ \mathrm{A} \\ \mathrm{C} \\ \mathrm{B} \end{pmatrix}=\begin{pmatrix} 4 \\ 16 \\ 36 \\ 64 \end{pmatrix}. \tag{39}$$

The solution of (39) with (36), as obtained by subroutine ELLP in ELLP5.FOR, agrees with the values of those quantities as given in (38) to within eight significant digits. Finally, using those results in (34), with $\mathrm{x}_1=0$, $\mathrm{y}_1=0$, gives D with comparable accuracy. The center point coordinates are given by $(\mathrm{x_c},\mathrm{y_c})=(2\mathrm{A}[\mathrm{Bb}-2\mathrm{a}]/\mathrm{T},\ 2[\mathrm{a}-2\mathrm{Ab}]/\mathrm{T})=(-20/7,44/7)$. A plot of E is shown in Figure 12 with its center and with the five specified points.

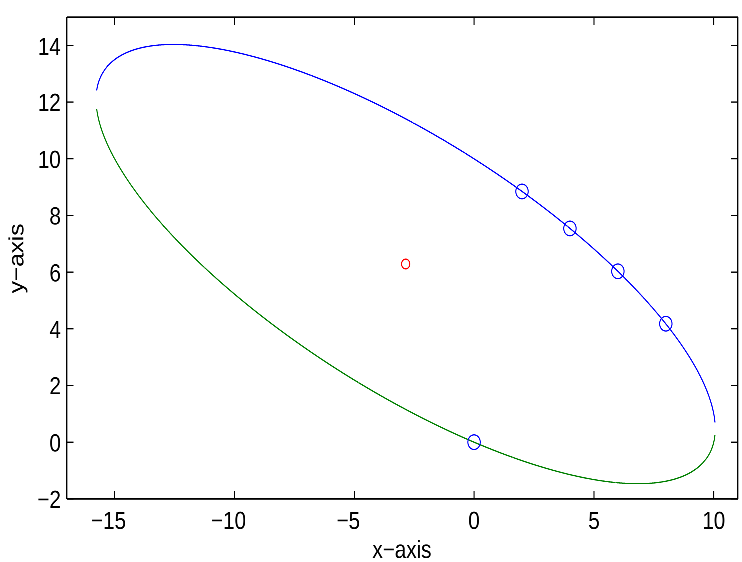

Figure 12. Plot of E with Five Specified Points in xyp

VI. FINDING COORDINATES OF A TARGET FROM TDOAs

The TDOAs are used to determine the coordinates (Tx,Ty) of a target point T, by using a pulse emitted by a transmitter.

Given a transmitter T1, located at (T1x, T1y) and three sensors, S1k, k=1,2,3 with locations (S1kx, S1ky). See Figure 13.

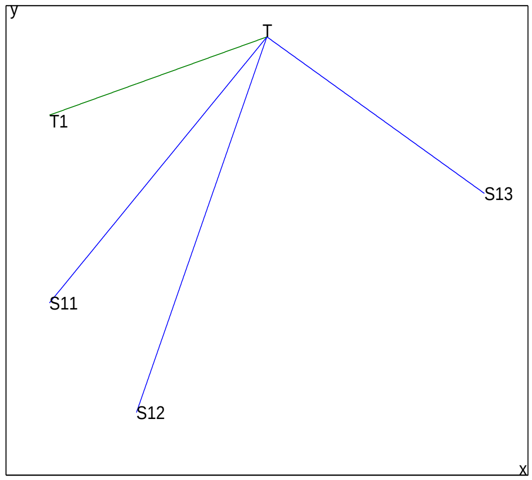

Figure 13. Multistatic Network with One Transmitter and Three Sensors

A pulse P1 is sent from T1 to T and received from T at S11, S12, S13. Let the time taken for each of these paths be denoted by T1T, TS11, TS12, and TS13, respectively. Then we can define times of arrival TOA11 and TOA12 by

$$\mathrm{TOA11} \equiv \mathrm{T1T} + \mathrm{TS11}, \tag{40}$$

$$\mathrm{TOA12} \equiv \mathrm{T1T} + \mathrm{TS12}. \tag{41}$$

Subtracting (40) from (41) defines TDOA1,

$$\mathrm{TDOA1} \equiv \mathrm{TOA12} - \mathrm{TOA11} = \mathrm{TS12} - \mathrm{TS11}. \tag{42}$$

Also,

$$\mathrm{TOA13} \equiv \mathrm{T1T} + \mathrm{TS13}. \tag{43}$$

Then, subtracting (40) from (43) defines TDOA2 accordingly

$$\mathrm{TDOA2} \equiv \mathrm{TOA13} - \mathrm{TOA11} = \mathrm{TS13} - \mathrm{TS11}. \tag{44}$$

We are now in position to achieve the objective of determining the target point coordinates (Tx,Ty). Let c denote the speed of sound in meters/second,[3] and note that

$$\mathrm{c} * \mathrm{TS1k} = \sqrt{(\mathrm{Tx} - \mathrm{S1kx})^2 + (\mathrm{Ty} - \mathrm{S1ky})^2}, \qquad \mathrm{k} = 1, 2, 3. \tag{45}$$

[3] $\mathrm{c} = 331.3 * \sqrt{1 + \mathrm{X}/273.15}, \quad \mathrm{X} = 5/9 * (\mathrm{F} - 32), \quad \mathrm{F} \equiv$ Degrees Fahrenheit.

Then from (42) and (44) we have two nonlinear equations to solve for (Tx,Ty), namely

$$\mathrm{c} * \mathrm{TDOA1} = \sqrt{(\mathrm{Tx} - \mathrm{S12x})^2 + (\mathrm{Ty} - \mathrm{S12y})^2} - \sqrt{(\mathrm{Tx} - \mathrm{S11x})^2 + (\mathrm{Ty} - \mathrm{S11y})^2}, \quad (46)$$

$$\mathrm{c} * \mathrm{TDOA2} = \sqrt{(\mathrm{Tx} - \mathrm{S13x})^2 + (\mathrm{Ty} - \mathrm{S13y})^2} - \sqrt{(\mathrm{Tx} - \mathrm{S11x})^2 + (\mathrm{Ty} - \mathrm{S11y})^2}. \quad (47)$$

Note that these equations are independent of the location of T1.

In Fortran file EXLAST2.FOR, the subroutine HBRD [5] is used to solve (46) and (47) for Tx and Ty. A numerical example, assuming acoustical instruments, with distances in meters and velocity in meters/seconds, follows. Input is:
TEMP = 70°F, c = 343.8644,
S11x = 3D3, S11y = 0, S12x = 5.71D3, S12y = 3.525D3, S13x = 1D3, S13y = 4D3, [4]
TDOA1 = 1D − 1, TDOA2 = 1.52D − 1.
With an initial guess of Tx = Ty = 1D3, one obtains, as output, the final target coordinates:
Tx = 3.2466935D3, Ty = 2.5890217D3.

A second numerical example is based on Jack Carr's experimental setup, where a constraint is imposed, i.e.,

$$\mathrm{Ty} = \mathrm{a} * \mathrm{Tx} + \mathrm{b}, \quad (\mathrm{a}, \mathrm{b} \text{ given}). \quad (48)$$

In this case only two sensors are required resulting in one TDOA.
TEMP = 70°F, c = 343.8644, a = b = 1,
S11x = 3D3, S11y = 0, S12x = 5.71D3, S12y = 3.525D3,
TDOA1 = 8D − 2.
Using Fortran subroutine HBRD in EXCONST.FOR, with an initial guess of Tx = 1D3, one obtains the final target coordinates:
Tx = 2.87598D3, Ty = 2.876.98D3.

In the third and final example, RF signals are transmitted that travel at the speed of light, namely c = 2.9979248D5 km/sec; the analysis would hold for acoustic signals as well. Three sensors and one transmitter make up the network to locate a target in 3-space with a constraint. The Earth is assumed to be spherical with radius $\mathrm{R_E} = 6378.137$ km.

The method of solution to determine the target coordinates $(\mathrm{x}, \mathrm{y}, \mathrm{z})$[5] is different than the one used in the previous examples. It is described in the Ho-Chan paper [4].

Specifically, our objective is to develop Fortran 95 software that is based on one of the results of the Ho-Chan paper, [4]. The software will output the coordinates of a target from the use of three sensors, whose three-dimensional position coordinates are known, with the constraint that the normal distance from the center of the Earth to the target, r, is also

[4]Numerical values are specified in Fortran notation. For example, $2.3\mathrm{D} - 3 \equiv 2.3 * 10^{-3} = .0023$. All computations are carried out in double precision.

[5]Hereafter the notation of [4] is used, including bold upper case letters to denote matrices and bold lower case letters to denote vectors.

known. As noted in [4], such a constraint has practical application as, for example, a target sitting on the Earth's surface. The constraint allows the location of the target with only two measurements of TDOAs.

The location of the target is denoted by $\mathbf{u} = [x, y, z]^T$ and the locations of the three sensors are specified by $\mathbf{s}_k = [x_k, y_k, z_k]^T, \quad k = 1, 2, 3$. Thus, the distance between target and sensor k is:

$$r_k = |\mathbf{s}_k - \mathbf{u}| = \sqrt{(x_k - x)^2 + (y_k - y)^2 + (z_k - z)^2}, \quad k = 1, 2, 3. \tag{49}$$

To clarify the meaning of TDOA here, consider an Earth station from which a signal, at arbitrary time=0, moving at the speed of light, c, travels to the target and the reflected signal then travels to each of the three sensors (It is assumed that not any three of $\{\mathbf{0}, \mathbf{s_1}, \mathbf{s_2}, \mathbf{s_3}\}$ lie on a straight line; the Earth's center is at $\mathbf{0}$.). Let TOA_k denote the time of arrival of the signal at sensor k. Then let
$d_{k,1} = TDOA_{k,1} = TOA_k - TOA_1, \;\; k = 2, 3$, so that

$$r_{k,1} = cd_{k,1} = r_k - r_1, \quad k = 2, 3. \tag{50}$$

Note that the constraint can also be written as

$$\mathbf{u}^T\mathbf{u} = (R_E + D)^2 = r^2, \tag{51}$$

where $R_E \equiv \text{spherical Earth's radius}, \;\; D \equiv \text{distance above the Earth along } \mathbf{u}$.

From this point, we proceed mathematically, following [4], toward the objective of finding $\mathbf{u}$, given: $\mathbf{s}_k$, $TDOA_{2,1}$, $TDOA_{3,1}$, and r. Rewriting (50) as $r_{k,1} + r_1 = r_k$ and squaring both sides of this relation gives

$$r_{k,1}^2 + 2r_{k,1}r_1 + r_1^2 = r_k^2 = (\mathbf{s}_k - \mathbf{u})^T(\mathbf{s}_k - \mathbf{u}) = r^2 + \mathbf{s}_k^T\mathbf{s}_k - 2\mathbf{s}_k^T\mathbf{u}, \quad k = 2, 3, \tag{52}$$

where r_1 from (49) gives

$$r_1^2 = r^2 + \mathbf{s}_1^T\mathbf{s}_1 - 2\mathbf{s}_1^T\mathbf{u}. \tag{53}$$

Replacing $\mathbf{r}_1^2$ on the left side of (52) by (53) gives

$$r_{k,1}^2 + 2r_{k,1}r_1 = \mathbf{s}_k^T\mathbf{s}_k - \mathbf{s}_1^T\mathbf{s}_1 - 2(\mathbf{s}_k - \mathbf{s}_1)^T\mathbf{u}, \quad k = 2, 3. \tag{54}$$

Equations (53) and (54) represent a set of linear equations for $\mathbf{u}$ in terms of the variable r_1. In matrix notation we have

$$\mathbf{G}\mathbf{u} = \mathbf{h}, \tag{55}$$

where

$$\mathbf{G} = -2\begin{bmatrix} \mathbf{s}_1^T \\ \mathbf{s}_2^T - \mathbf{s}_1^T \\ \mathbf{s}_3^T - \mathbf{s}_1^T \end{bmatrix}, \tag{56}$$

$$\mathbf{h} = \mathbf{H}\begin{bmatrix}1\\ \mathrm{r}_1\\ \mathrm{r}_1^2\end{bmatrix} \equiv \begin{bmatrix}-\mathrm{r}^2-\mathbf{s}_1^{\mathrm{T}}\mathbf{s}_1 & 0 & 1\\ \mathrm{r}_{2,1}^2-\mathbf{s}_2^{\mathrm{T}}\mathbf{s}_2+\mathbf{s}_1^{\mathrm{T}}\mathbf{s}_1 & 2\mathrm{r}_{2,1} & 0\\ \mathrm{r}_{3,1}^2-\mathbf{s}_3^{\mathrm{T}}\mathbf{s}_3+\mathbf{s}_1^{\mathrm{T}}\mathbf{s}_1 & 2\mathrm{r}_{3,1} & 0\end{bmatrix}\begin{bmatrix}1\\ \mathrm{r}_1\\ \mathrm{r}_1^2\end{bmatrix}. \tag{57}$$

As noted in [4], the existence of $\mathbf{G}^{-1}$ requires that any three of $(0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$ cannot lie on a straight line. In addition, the three sensors should have sufficient spatial separation to avoid ill-conditioning of $\mathbf{G}$. Hence, from (55),

$$\mathbf{u} = \mathbf{G}^{-1}\mathbf{h} = \mathbf{Q}\,[\,1, \mathrm{r}_1, \mathrm{r}_1^2\,]^{\mathrm{T}}, \tag{58}$$

where

$$\mathbf{Q} \equiv \mathbf{G}^{-1}\mathbf{H}. \tag{59}$$

Substituting into (51), we have

$$\mathbf{u}^{\mathrm{T}}\mathbf{u} - \mathrm{r}^2 = [\,1, \mathrm{r}_1, \mathrm{r}_1^2\,]\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\,[\,1, \mathrm{r}_1, \mathrm{r}_1^2\,]^{\mathrm{T}} - \mathrm{r}^2 = 0. \tag{60}$$

which represents a quartic polynomial in r_1. Let $\mathbf{P}$ denote a symmetric matrix such that

$$\mathbf{P} \equiv \begin{bmatrix}\mathrm{p}_{11} & \mathrm{p}_{12} & \mathrm{p}_{13}\\ \mathrm{p}_{12} & \mathrm{p}_{22} & \mathrm{p}_{23}\\ \mathrm{p}_{13} & \mathrm{p}_{23} & \mathrm{p}_{33}\end{bmatrix} \equiv \mathbf{Q}^{\mathrm{T}}\mathbf{Q} \tag{61}$$

Expanding (60) and using (57), (59), and (61), we obtain the coefficients of the quartic F. Thus,

$$\mathrm{F}(\mathrm{r}_1) \equiv (\,\mathrm{p}_{11} - \mathrm{r}^2\,) + 2\,\mathrm{p}_{12}\,\mathrm{r}_1 + (\,2\,\mathrm{p}_{13} + \mathrm{p}_{22}\,)\mathrm{r}_1^2 + 2\,\mathrm{p}_{23}\,\mathrm{r}_1^3 + \mathrm{p}_{33}\,\mathrm{r}_1^4 = 0 \tag{62}$$

We are interested in the positive roots of F. Using one such root in (58) gives a set of possible target coordinates. In case there is more than one positive root, as pointed out in [4], extraneous roots will not satisfy (50). However, it still can happen that more than one positive root satisfies (50), in which case the user must decide by other means which is the meaningful one for his application. The numerical example that follows reflects such a case.

The file CHAN.FOR uses four subroutines from the NSWC Library of Mathematical Subroutines, [5]:

(1) CROUT (Input: $\mathbf{G}$, output: $\mathbf{G^{-1}}$ stored in G.)
(2) MTMS (Multiplies $\mathbf{G^{-1}}, \mathbf{H}$, stores result in Q.)
(3) TMPROD (Input: $\mathbf{Q}$, output: $\mathbf{Q^T Q}$ stored in QT.)
(4) DRPOLY (Input: Coefficients of polynomial F (elements of QT, see (62)) stored in T1, output: real parts of roots of F in array ZR and imaginary parts in array ZI.)

We use c (speed of light)= 2.9979248D5 km/sec, $\mathrm{R_E}$ (radius of the Earth)= 6378.137 km.

Name of Fortran 95 file: Chan.for (Contains main program plus the four subroutines mentioned above, including their supporting routines.)

Input: D (distance above the Earth of the target)= 1 km

$$\begin{aligned}
\mathbf{s}_1 &= [0.0 \quad 0.0 \quad 6378.137]^{\mathrm{T}} \\
\mathbf{s}_2 &= [3189.069 \quad 3189.069 \quad 4510.024]^{\mathrm{T}} \\
\mathbf{s}_3 &= [2761.814 \quad 4783.603 \quad 3189.069]^{\mathrm{T}} \\
\mathrm{TDOA}_{1,2} &= 2\mathrm{D}-4 \\
\mathrm{TDOA}_{1,3} &= 5\mathrm{D}-4 \\
\mathrm{r} &= 6379.137
\end{aligned} \tag{63}$$

Intermediate Results:

$$\begin{aligned}
\mathbf{G} &= \begin{bmatrix} -7.81071\mathrm{D}-13 & 0.00000\mathrm{D}0 & -1.27563\mathrm{D}4 \\ -6.37814\mathrm{D}3 & -6.37814\mathrm{D}3 & 3.73623\mathrm{D}3 \\ -5.52363\mathrm{D}3 & -9.56721\mathrm{D}3 & 6.37814\mathrm{D}3 \end{bmatrix} \\
\mathbf{H} &= \begin{bmatrix} -8.13740\mathrm{D}7 & 0.00000\mathrm{D}0 & 1.00000\mathrm{D}0 \\ 3.59502\mathrm{D}3 & 1.19917\mathrm{D}2 & 0.00000\mathrm{D}0 \\ 2.24689\mathrm{D}4 & 2.99792\mathrm{D}2 & 0.00000\mathrm{D}0 \end{bmatrix} \\
\mathbf{G}^{-1} &= \begin{bmatrix} 1.50016\mathrm{D}-5 & -3.70959\mathrm{D}-4 & 2.47306\mathrm{D}-4 \\ -6.09230\mathrm{D}-5 & 2.14173\mathrm{D}-4 & -2.47306\mathrm{D}-4 \\ -7.83928\mathrm{D}-5 & 2.27139\mathrm{D}-20 & -1.51426\mathrm{D}-20 \end{bmatrix} \\
\mathbf{Q} &= \begin{bmatrix} -1.21652\mathrm{D}3 & 2.96562\mathrm{D}-2 & 1.50016\mathrm{D}-5 \\ 4.95277\mathrm{D}3 & -4.84574\mathrm{D}-2 & -6.09230\mathrm{D}-5 \\ 6.37914\mathrm{D}3 & -1.81586\mathrm{D}-18 & -7.83928\mathrm{D}-5 \end{bmatrix} \\
\mathbf{Q}^{\mathbf{T}}\mathbf{Q} &= \begin{bmatrix} 6.67032\mathrm{D}7 & -2.76076\mathrm{D}2 & -8.20066\mathrm{D}-1 \\ -2.76076\mathrm{D}2 & 3.22761\mathrm{D}-3 & 3.39706\mathrm{D}-6 \\ -8.20066\mathrm{D}-1 & 3.39706\mathrm{D}-6 & 1.00821\mathrm{D}-8 \end{bmatrix} \\
& \mathrm{Det} = -3.28991\mathrm{D}11 \quad (\text{Determinant of } \mathbf{G})
\end{aligned} \tag{64}$$

$$\mathrm{F}(\mathrm{r}_1) = \sum_{\mathrm{k}=1}^{5} \mathrm{T1}(\mathrm{k})\, \mathrm{r}_1^{5-\mathrm{k}} = 0, \tag{65}$$

where

$$\begin{aligned}
\mathrm{T1}(1) &= 1.00821\mathrm{D}-8 \\
\mathrm{T1}(2) &= 6.79413\mathrm{D}-6 \\
\mathrm{T1}(3) &= -1.63690\mathrm{D}0 \\
\mathrm{T1}(4) &= -5.52151\mathrm{D}2 \\
\mathrm{T1}(5) &= 2.60098\mathrm{D}7.
\end{aligned} \tag{66}$$

With F as given in (65), DRPLOY finds the roots:

$$\begin{aligned}
\mathrm{ZR}(1) &= 4.05816\mathrm{D}3, & \mathrm{ZI}(1) &= 0.0 \\
\mathrm{ZR}(2) &= -4.39528\mathrm{D}3, & \mathrm{ZI}(2) &= 0.0 \\
\mathrm{ZR}(3) &= 1.18592\mathrm{D}4, & \mathrm{ZI}(3) &= 0.0 \\
\mathrm{ZR}(4) &= -1.21960\mathrm{D}4, & \mathrm{ZI}(4) &= 0.0.
\end{aligned} \tag{67}$$

Output:

We are only interested in the positive roots ZR(1) and ZR(3). Noting from (58) that

$$\mathbf{u} = \mathbf{Q}[1, \mathrm{r}_1, \mathrm{r}_1^2]^{\mathrm{T}} \tag{68}$$

there are two solutions, namely

$$\mathbf{u} = [-8.49113\mathrm{D}2, \;\; 3.75280\mathrm{D}3, \;\; 5.08811\mathrm{D}3]^{\mathrm{T}} \tag{69}$$

$$\mathbf{u} = [1.24501\mathrm{D}3, \;\; -4.19015\mathrm{D}3, \;\; -4.64607\mathrm{D}3]^{\mathrm{T}}. \tag{70}$$

Both solutions, using (49), satisfy (50) with

$$\mathrm{r}_{2,1} = 59.95848, \quad \mathrm{r}_{3,1} = 149.8962. \tag{71}$$

Thus, neither positive root of (65) is extraneous and, therefore, both solutions hold.

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APPENDIX A

LIST OF FORTRAN 95 SUBPROGRAMS

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APPENDIX A

LIST OF FORTRAN 95 SUBPROGRAMS

1. MCARR.FOR or MCARR1.FOR−File containing programs to generate L−lines from an M-set, based on the Hough Transformation.
   (a)GUVM−Subroutine that generates an M-set of points to test HSORT1.
   (b)HSORT1−Subroutine that finds L−lines in an M-set of points.
   (c)MATPLO−Generates a Matlab M file for plotting results from HSORT1.
   (d)HSORTI−Sorting routine from Mathlib, [5] of the main text.

2. DH.FOR−File containing subprograms to generate L−arcs from an M-set.
   (a)DGXY−Subroutine generating an M-set of points for testing subroutine CIRSOR.
   (b)CIR1−Subroutine that finds parameters defining a circle from three M-points.
   (c)CIRSOR−Subroutine that finds L−arcs from an M-set of points.
   (d)MATPLX−Generates an Matlab M file for plotting results from CIRSOR.
   (e)LE−Supporting routine for CIRSOR. See Appendix B.
   (f)LCAL1−Supporting routine for LE.
   (g)RNORM−Normal random number generating routine used in DGXY, taken from [5] of the main text.
   (h)RCIR1−Supporting routine for RNORM.

3. PTCIRC.FOR−File containing subprograms to generate L–lines, each with an attached circular arc at one endpoint.
   (a)GXYC−Generates an M-set of points to test HSORT1 together with HSORT2.
   (b)HSORT1−See above.
   (c)HSORT2−From an M-set, finds a circular arc attached to an endpoint of an L−line.
   (d)GUV−Subroutine that generates the parameters of an L−line of an M-set.
   (e)GUVR−Subroutine that generates the parameters of a circular arc attached to an L−line.
   (f)CIR−Subroutine, essentially the same as CIR1 above.
   (g)MATPLO−Generates a Matlab M file for plotting results from GXYC.
   (h)RNORM−See above.
   (i)RCIR1−See above.

4. ELLP5.FOR−File containing subprograms to find parameters defining an ellipse, given five points in the xy-plane.
   (a)ELLPT−Given ellipse parameters and ellipse coordinates $x1, y1, x2$, ELLPT finds two y-coordinates on the ellipse with x2 as the x-coordinate.

(b)ELLP–Given five points in the plane, ELLP determines the ellipse, E, with those points on E, by outputting the parameters defining E.
(c)CROUT–Subroutine that solves a linear system, with real coefficients, by the Crout procedure. Contained in [5] of the main text.
(d)ROTA–Gives the angle of rotation of the xy-axes so that the xy term in (34) of the main text is eliminated. The output also includes the parameters defining E in the rotated coordinates.
(e)RNORM–See above.
(f)RCIR1–See above.

5. EXLAST2.FOR–File containing subprograms for determining target coordinates from a specified set of sensors and a transmitter.
   (a)HBRD–Subroutine that solves a set of nonlinear functions. Contained in [5] of the main text.

6. CHAN.FOR–File containing subprograms for using the Chan method to determine three-dimensional target coordinates using three sensors and a constraint.
   (a)CROUT–See above.
   (b)MTMS–Subroutine, from [5] of the main text, for multiplying two real matrices.
   (c)TMPROD–Subroutine, from [5] of the main text, to produce $A^T B$ given real matrices A and B.
   (d)DRPOLY–Subroutine, from [5] of the main text, to produce the roots of a polynomial with real coefficients.

APPENDIX B

A RESULT USED IN SUBROUTINE LCAL1 CALLED BY SUBROUTINE LE

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APPENDIX B

A RESULT USED IN SUBROUTINE LCAL1 CALLED BY SUBROUTINE LE

This analysis is used in subroutine LE of file HCIR.FOR and the double precision version DHCIR.FOR. It is also used in the latest circular arc routine DCIR3.FOR. LCAL1 replaces the subroutine IJKL, which is much slower.

We have the loops:

```
L = 0
DO 5 I=1, N-2  !N IS TOTAL NO. OF (X,Y) POINTS. SEE ROUTINE CIRSOR
DO 5 J=I+1, N-1
DO 5 K=J+1, N
L = L + 1
5 CONTINUE
```

PROBLEM: Given N, I, J, K. Find L without running through 3 LOOPS.
For example: N=10, I=2, J=5, K=6; FIND L. It is L=50.

We use:

$$\sum_{m=1}^{n} m = n(n+1)/2, \qquad \sum_{m=1}^{n} m^2 = n(n+1)(2n+1)/6. \tag{B-1}$$

We first find the contribution to L from I; call it L_I. Then

$$L_I = \frac{1}{2}\sum_{m=1}^{\hat{I}}(N-m)(N-m-1), \qquad \hat{I} = I - 1. \tag{B-2}$$

$$2\,L_I = \hat{I}\,N^2 - \hat{I}\,N\,(\hat{I}+1) + \frac{1}{6}\,\hat{I}\,(\hat{I}+1)(2\,\hat{I}+1) - \hat{I}\,N + \frac{1}{2}\,\hat{I}\,(\hat{I}+1), \tag{B-3}$$

$$2\,L_I = \hat{I}\,N^2 - \hat{I}\,N\,(\hat{I}+2) + \frac{1}{6}\,\hat{I}\,(\hat{I}+1)(2\,\hat{I}+1) + \frac{1}{2}\,\hat{I}\,(\hat{I}+1), \tag{B-4}$$

$$2\,L_I = \hat{I}\,N\,(N-I-1) + \frac{1}{3}\,\hat{I}\,I\,(I+1), \tag{B-5}$$

$$L_I = \frac{\hat{I}\,[\,3\,N\,(N-I-1) + I\,(I+1)]}{6}. \tag{B-6}$$

For a fixed I, the contribution to L from J and K, denoted by L_{JK}, is given by

$$L_{JK} = \sum_{n=I+1}^{J-1}(N-n) + K - J, \tag{B-7}$$

$$L_{JK} = (J-I-1)N - J(J-1)/2 + I(I+1)/2 + K - J, \tag{B-8}$$

$$L_{JK} = [(J-I-1)(2N-I-J)]/2 + K - J. \tag{B-9}$$

Hence

$$L = L_I + L_{JK}. \tag{B-10}$$

Completing the numerical example above, we get for $N = 10$, $I = 2$, $J = 5$, $K = 6$; $L_I = 36$, $L_{JK} = 14$, $L = 50$.

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W11 STEVEN ANDERSON	1
W12 WILLIAM ORMSBY	1
W21 DAVID CLAWSON	1
W30 DR ARMIDO DIDONATO	2
W30 EMMANUEL SKAMANGAS	1
W31 DR ROBERT MCDEVITT	1
W31 DR DEMETRIOS SERAKOS	1
W60 ROBERT HILL	1
Z31 TECHNICAL LIBRARY	3

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